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1998 J. Phys. A: Math. Gen. 31 7561

(http://iopscience.iop.org/0305-4470/31/37/014)

J. Phys. A: Math. Gen. **31** (1998) 7561–7575. Printed in the UK PII: S0305-4470(98)89686-8

Exact solutions of the Dirac equation with harmonic pseudoscalar, scalar or electric potential

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Received 2 December 1997, in final form 25 March 1998

Abstract. A new class of exact solutions is obtained in explicit form for the Dirac equation with a pseudoscalar or scalar, or electric static potential. The potential may be an arbitrary harmonic function the gradient squared of which is a constant. It is shown that the set of such potentials is sufficiently ample. The simplest example is the linear function $\phi(\boldsymbol{x}) = ax_1 + bx_2 + cx_3 + d$. Another example is the function $\phi(\boldsymbol{x}) = \psi(z) + cx_3$, where ψ is an arbitrary analytic function depending on the complex variable $z = x_1 + \mathrm{i}x_2$. The solutions are obtained using the technique of biquaternionic projection operators which itself is interesting since its possible applications are not limited to the situation considered in this paper.

1. Introduction

Exact solutions of the Dirac equation are of special interest, since in many cases (due to the general facts from functional analysis and the theory of partial differential equations) it is possible to derive some qualitative conclusions about the behaviour of the quantum system or even solve the corresponding Cauchy or boundary value problem. In particular, all corresponding Green's functions were constructed using exact solutions of the Dirac equation. Of course, there exist dozens of works on the topic. The reader is referred to the encyclopaedic monograph [1] for a bibliography and review of known exact solutions of the Dirac equation up to the late 1980s.

In this work we consider the following Dirac equations: (a) with pseudoscalar potential

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + \mathrm{i}m + \gamma_0\gamma_5\phi(\boldsymbol{x})\right]\Phi(t,\boldsymbol{x}) = 0 \tag{1}$$

(b) with scalar potential

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + \mathrm{i}m + \phi(\boldsymbol{x})\right]\Phi(t,\boldsymbol{x}) = 0 \tag{2}$$

and (c) with electric potential

$$\left[\gamma_0\partial_t - \sum_{k=1}^{3}\gamma_k\partial_k + \mathrm{i}m + \mathrm{i}\gamma_0\phi(\boldsymbol{x})\right]\Phi(t,\boldsymbol{x}) = 0 \tag{3}$$

where the Dirac matrices have the standard [8, 22] Dirac–Pauli form

$$\gamma_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma_1 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_2 := \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \gamma_5 := \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and where $\partial_t := \frac{\partial}{\partial t}$, $\partial_k := \frac{\partial}{\partial x_k}$, $m \in \mathbb{R}$, ϕ is a complex-valued function, and Φ is a $\mathbb{C}^4$-valued function.

We shall construct a class of time-harmonic solutions of these equations: $\Phi(t, \boldsymbol{x}) = q(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t}$, where $\omega \in \mathbb{R}$ and q is a $\mathbb{C}^4$-valued function depending on $\boldsymbol{x} = (x_1, x_2, x_3)$. For example, in the case of pseudoscalar potential (equation (1)) we have the following equation for q:

$$\mathbb{D}^{ps}_{\omega,m} q(\boldsymbol{x}) := \left[\mathrm{i}\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}m + \gamma_0\gamma_5\phi(\boldsymbol{x})\right] q(\boldsymbol{x}) = 0. \tag{4}$$

The procedure proposed here consists of the following. First, we rewrite the operator $\mathbb{D}^{ps}_{\omega,m}$ and equation (4) in the biquaternionic form with the aid of a certain special matrix transform introduced in [14] (see also [17, section 12]). Then, applying the technique of biquaternionic projection operators developed in [15], we construct a class of exact solutions of (4) in biquaternionic form. In order to simplify the exposition we consider first the massless, static case ($m = 0$, $\omega = 0$) (in section 3); then the general case in quaternionic form (in section 4), and then using these results we find a class of solutions to (1) in the general situation and traditional form (in section 5). To do this we have to distinguish two different cases: (1) $\omega^2 \neq m^2$ and (2) $\omega^2 = m^2$ which, in the quaternionic form, correspond to the situations where the biquaternionic parameter α generated by ω and m (1) is not a zero divisor and (2) is a zero divisor. After the class of solutions is obtained in the quaternionic form we rewrite it in the usual form and the results of this paper can be verified by a direct substitution of the proposed solutions to the corresponding Dirac equation. In section 6 using a similar procedure we construct some exact solutions to (2) in the static massive case. In section 7 we deal with the electric potential in the massless case (but $\omega \neq 0$). A class of exact solutions in explicit form is also obtained here.

Of course, the application of complex quaternions and even more so, of Clifford algebras to Dirac equation theory is not new, but has been exploited by many authors (such as [4–6, 10, 21, 23] and many others). It is an approach that allows one to use the natural advantages of the algebraic structures to simplify all the calculations. We would like to emphasize that in our opinion the tools of quaternionic analysis are among the most appropriate for studying the Dirac operator. The fact is that the Clifford algebra generated by γ-matrices has dimension 16, and it does not make sense to use an algebra of 16 dimensions if the number of equations under consideration is four. Similar arguments led Sommerfeld to pose the following problem: to rewrite the Dirac equation in a form in which the rank of the algebra of the involved matrices coincides with the number of components of the wavefunction. A review of results in this direction can be found in [5, chapter 4]. Thus, representation of the Dirac equation in terms of complex quaternions is in some sense the optimum although some other interesting possibilities to reduce the dimension of the corresponding algebra should be mentioned, namely, the real Dirac algebra (e.g. see [13]) and the manner of using the Pauli algebra [2, 3]. Nevertheless the aim of this work is not to show the advantages of quaternionic analysis over other techniques, but to present some new results in the theory of exact solutions of Dirac's equation. We start with the Dirac equation written in traditional form and conclude with its solutions also in traditional form, using complex quaternions only to obtain the results.

2. Preliminaries

We denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (= biquaternions). The elements of $\mathbb{H}(\mathbb{C})$ are represented in the form $\rho = \sum_{k=0}^{3} \rho_k i_k$, where $\{\rho_k\} \subset \mathbb{C}$, i_0 is the unit and i_k, $k = \overline{1,3}$ are standard quaternionic imaginary units: $i_k^2 = -1$, $k = 1, 2, 3$; $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$, $i_3 i_1 = -i_1 i_3 = i_2$. We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition i commutes with i_k. We will use also the vector representation of $\rho \in \mathbb{H}(\mathbb{C})$: $\rho = \mathrm{Sc}(\rho) + \mathrm{Vec}(\rho)$, where $\mathrm{Sc}(\rho) := \rho_0$ and $\mathrm{Vec}(\rho) := \boldsymbol{\rho} = \sum_{k=1}^{3} \rho_k i_k$. The complex quaternions of the form $\rho = \boldsymbol{\rho}$ are called purely vectorial and identified with the vectors from $\mathbb{C}^3$. The quaternion $\overline{\rho} := \rho_0 - \boldsymbol{\rho}$ is called conjugate to ρ.

Let us denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. Note that

$$\rho \in \mathfrak{S} \iff \rho\overline{\rho} = 0 \iff \rho^2 = 2\rho_0\rho \iff \rho_0^2 = (\boldsymbol{\rho})^2 \tag{5}$$

(see [17, p 28]). As usual zero is not included to $\mathfrak{S}$.

We shall consider the $\mathbb{H}(\mathbb{C})$-valued functions given in a domain $\Omega \subset \mathbb{R}^3$. On the set $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the well known Moisil–Theodoresco operator is defined by the expression $\boldsymbol{D} := \sum_{k=1}^{3} i_k \partial_k$, which was introduced in [18, 19] (see also, e.g. [7, 9, 11, 12]). Define $\tilde{q}(\boldsymbol{x}) := q(x_1, x_2, -x_3)$. The domain $\tilde{G}$ is assumed to be obtained from the domain $G \subset \mathbb{R}^3$ by the reflection $x_3 \to -x_3$.

In [14] (see also [17, section 12, 16]) a map $\mathcal{A}$ was introduced which transforms a function $q : \tilde{G} \subset \mathbb{R}^3 \to \mathbb{C}^4$ into a function $\rho : G \subset \mathbb{R}^3 \to \mathbb{H}(\mathbb{C})$ by the rule

$$\rho = \mathcal{A}[q] := \tfrac{1}{2}[-(\tilde{q}_1 - \tilde{q}_2)i_0 + \mathrm{i}(\tilde{q}_0 - \tilde{q}_3)i_1 - (\tilde{q}_0 + \tilde{q}_3)i_2 + \mathrm{i}(\tilde{q}_1 + \tilde{q}_2)i_3].$$

Note that $\mathcal{A}$ is a $\mathbb{C}$-linear transform. The corresponding inverse transform is defined by the following equality:

$$\mathcal{A}^{-1}[\rho] = (-\mathrm{i}\tilde{\rho}_1 - \tilde{\rho}_2, -\tilde{\rho}_0 - \mathrm{i}\tilde{\rho}_3, \tilde{\rho}_0 - \mathrm{i}\tilde{\rho}_3, \mathrm{i}\tilde{\rho}_1 - \tilde{\rho}_2).$$

The transformations $\mathcal{A}$ and $\mathcal{A}^{-1}$ may be represented in a matrix form as follows:

$$\rho = \mathcal{A}[q] = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ -1 & 0 & 0 & -1 \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix}$$

and

$$q = \mathcal{A}^{-1}[\rho] = \begin{pmatrix} 0 & -\mathrm{i} & -1 & 0 \\ -1 & 0 & 0 & -\mathrm{i} \\ 1 & 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\rho}_0 \\ \tilde{\rho}_1 \\ \tilde{\rho}_2 \\ \tilde{\rho}_3 \end{pmatrix}.$$

We shall need some algebraic properties of this pair of transforms proved in [16]:

(1) $\mathcal{A}[\gamma_1\gamma_2\gamma_3\gamma_1\mathcal{A}^{-1}[\rho]] = i_1\rho$;
(2) $\mathcal{A}[\gamma_1\gamma_2\gamma_3\gamma_2\mathcal{A}^{-1}[\rho]] = i_2\rho$;
(3) $\mathcal{A}[\gamma_1\gamma_2\gamma_3\gamma_3\mathcal{A}^{-1}[\rho]] = -i_3\rho$;
(4) $\mathcal{A}[\gamma_1\gamma_2\gamma_3\gamma_0\mathcal{A}^{-1}[\rho]] = \rho i_1$;

(5) $\mathcal{A}[\gamma_1\gamma_2\gamma_3\mathcal{A}^{-1}[\rho]] = -\mathrm{i}\rho i_2$.

Let us consider the following biquaternionic operator:

$$R_\alpha^{ps} := \boldsymbol{D} + \xi(\boldsymbol{x})I + \mathcal{M}^\alpha$$

where I is the identity operator, ξ is a complex-valued function, α is a constant complex quaternion, and $\mathcal{M}^\alpha$ is the operator of multiplication by α from the right-hand side: $\mathcal{M}^\alpha f := f\alpha$. The operator R_α^{ps} is equivalent to the operator $\mathbb{D}_{\omega,m}^{ps}$ in the following sense. Let $\alpha = -(\mathrm{i}\omega i_1 + m i_2)$, $\xi = -\mathrm{i}\tilde{\phi}$. Then

$$R_\alpha^{ps} = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}_{\omega,m}^{ps}\mathcal{A}^{-1}. \tag{6}$$

The proof of this equality is a direct corollary of the algebraic properties (1)–(5). of the transform $\mathcal{A}$. In other words a function q belongs to $\ker\mathbb{D}_{\omega,m}^{ps}(G)$ iff $u := \mathcal{A}[q] \in \ker R_\alpha^{ps}(\tilde{G})$. Thus, to find the solutions of (4) we first find the solutions of the equation

$$R_\alpha^{ps} u = 0 \tag{7}$$

using the algebraic advantages of complex quaternions as opposed to the Dirac matrices and then after applying the transform $\mathcal{A}^{-1}$ to u write down the corresponding solutions of (4). For the cases of scalar and electric potentials we shall show equalities similar to (7) in section 6 and section 7 respectively. Now we start with the case $\alpha = 0$.

3. Exact solutions of the equation $R_0^{ps}u = 0$

In this section we use the technique of projection operators introduced in [15]. Let us consider the equation

$$(\boldsymbol{D} + \xi(\boldsymbol{x}))u(\boldsymbol{x}) = 0 \tag{8}$$

in some domain $\tilde{G} \subset \mathbb{R}^3$ which, in particular, may coincide with $\mathbb{R}^3$. Let us consider the eikonal equation

$$(\operatorname{grad}\mu)^2 = \xi^2. \tag{9}$$

Note that here and in what follows a vector squared is understood in the quaternionic sense:

$$(\boldsymbol{h})^2 = (h_1 i_1 + h_2 i_2 + h_3 i_3)(h_1 i_1 + h_2 i_2 + h_3 i_3) = -\langle \boldsymbol{h}, \boldsymbol{h}\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product. Thus,

$$(\operatorname{grad}\mu)^2 = -\langle \operatorname{grad}\mu, \operatorname{grad}\mu\rangle.$$

Denote $\boldsymbol{g} := \operatorname{grad}\mu$. Then for any solution μ of (9) we have

$$(\xi(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}))(\xi(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})) = 0 \qquad \forall \boldsymbol{x} \in \tilde{G}$$

that is, the functions $\xi + \boldsymbol{g}$ and $\xi - \boldsymbol{g}$ are conjugate zero divisors and, consequently, generate a pair of mutually complementary, orthogonal projection operators:

$$Q^\pm := \frac{1}{2\xi}(\xi \pm \boldsymbol{g})I.$$

The operator $\boldsymbol{D} + \xi I$ can be rewritten in the form

$$\boldsymbol{D} + \xi I = Q^+(\boldsymbol{D} + \boldsymbol{g}I) + Q^-(\boldsymbol{D} - \boldsymbol{g}I) = (\boldsymbol{D} + \boldsymbol{g})Q^+ + (\boldsymbol{D} - \boldsymbol{g})Q^-$$

or alternatively

$$\boldsymbol{D} + \xi I = Q^+\eta\boldsymbol{D}\eta^{-1}I + Q^-\eta^{-1}\boldsymbol{D}\eta I = \eta\boldsymbol{D}\eta^{-1}Q^+ + \eta^{-1}\boldsymbol{D}\eta Q^-$$

where $\eta = \mathrm{e}^{-\mu}$ and $\boldsymbol{g} = \operatorname{grad}\mu = -\frac{\operatorname{grad}\eta}{\eta}$.

Thus, equation (8) is equivalent to the system

$$Q^{+}\eta \boldsymbol{D}\eta^{-1}u = 0 \tag{10}$$

$$Q^{-}\eta^{-1}\boldsymbol{D}\eta u = 0. \tag{11}$$

The general solution of each of these two equations can be immediately obtained in integral form (see [15]). The main problem is to find the intersection

$$\ker(Q^{+}\eta \boldsymbol{D}\eta^{-1}I)\bigcap(Q^{-}\eta^{-1}\boldsymbol{D}\eta I).$$

Let us assume that the function ξ satisfies the following two conditions:

$$(1)\ \Delta\xi = 0 \qquad (2)\ (\mathrm{grad}\,\xi)^2 = C^2 \tag{12}$$

where Δ is Laplacian and C is an arbitrary complex number different from zero. The class of functions satisfying the conditions (12) is sufficiently ample. The simplest example is the linear function $\xi = ax_1 + bx_2 + cx_3 + d$, where a, b, c, d are complex constants. Another example is the function $\xi(\boldsymbol{x}) = \zeta(z) + cx_3$, where ζ is an arbitrary analytic function (condition (1) is satisfied) depending on the complex variable $z = x_1 + \mathrm{i}x_2$. In this case

$$\mathrm{grad}\,\xi = \begin{pmatrix} \frac{\partial\zeta}{\partial z} \\ \mathrm{i}\frac{\partial\zeta}{\partial z} \\ c \end{pmatrix}$$

and condition (2) is also satisfied.

Due to condition (2) we are able to immediately construct the corresponding vector $\boldsymbol{g}$ satisfying the eikonal equation (9) as follows:

$$\boldsymbol{g} = \frac{\xi}{C}\,\mathrm{grad}\,\xi.$$

Then $\boldsymbol{g} = \mathrm{grad}\,\mu = -\frac{\mathrm{grad}\,\eta}{\eta}$, where $\mu = \frac{\xi^2}{2C}$ and $\eta = \mathrm{e}^{-\frac{\xi^2}{2C}}$. The projection operators have the form

$$Q^{\pm} := \frac{1}{2\xi}(\xi \pm \mathrm{grad}\,\mu)I = \frac{1}{2}\left(1 \pm \frac{1}{C}\,\mathrm{grad}\,\xi\right)I. \tag{13}$$

Then the functions $Q^{+}\mathrm{e}^{-\mu}$ and $Q^{-}\mathrm{e}^{\mu}$ as well as any function $f := Q^{+}\mathrm{e}^{-\mu}a^{+} + Q^{-}\mathrm{e}^{\mu}a^{-}$ are solutions of (8) [15]. Here a^{+} and a^{-} are arbitrary constant complex quaternions. For example, for the function $Q^{+}\mathrm{e}^{-\mu}$ we have

$$\begin{aligned}\boldsymbol{D}[Q^{+}\mathrm{e}^{-\mu}] &= \frac{1}{2}\boldsymbol{D}\left[\mathrm{e}^{-\frac{\xi^2}{2C}}\left(1 + \frac{1}{C}\,\mathrm{grad}\,\xi\right)\right] \\ &= -\frac{\xi}{2C}\,\mathrm{grad}\,\xi\cdot\mathrm{e}^{-\frac{\xi^2}{2C}}\left(1 + \frac{1}{C}\,\mathrm{grad}\,\xi\right) - \frac{1}{2C}\mathrm{e}^{-\frac{\xi^2}{2C}}\Delta\xi.\end{aligned}$$

In the first term we use condition (2) and the second term is zero due to condition (1). Thus, we obtain

$$\boldsymbol{D}[Q^{+}\mathrm{e}^{-\mu}] = -\frac{\xi}{2C}\mathrm{e}^{-\frac{\xi^2}{2C}}(\mathrm{grad}\,\xi + C) = -\xi Q^{+}\mathrm{e}^{-\mu}.$$

The inclusion $Q^{-}\mathrm{e}^{\mu} \in \ker(\boldsymbol{D} + \xi I)$ can be proved in the same manner. Moreover, the operators $\mathcal{M}^{a^{+}}$ and $\mathcal{M}^{a^{-}}$ commute with $\boldsymbol{D}$. Thus, we obtain the following statement.

Theorem 1. If the function ξ satisfies the conditions (12) in $\tilde{G} \subset \mathbb{R}^3$ then $f := Q^{+}\mathrm{e}^{-\mu}a^{+} + Q^{-}\mathrm{e}^{\mu}a^{-} \in \ker(\boldsymbol{D} + \xi I)(\tilde{G})$, where $\mu := \frac{\xi^2}{2C}$; the operators Q^{+} and Q^{-} are defined by (13); a^{+} and a^{-} are arbitrary constant complex quaternions.

Remark 1. Solutions of (8) similar to that of theorem 1 can be obtained for a more ample class of potentials ξ, for instance, for any ξ depending on a combination β of independent variables, where β satisfies (12). In this case the corresponding projection operators can be taken as follows $Q^{\pm} = \frac{1}{2}(1 \pm \frac{\mathrm{i}\,\mathrm{grad}\,\beta}{|\mathrm{grad}\,\beta|})$, and solutions of (8) are obtained in the same form as in theorem 1, where $\mu = \int \xi(\beta)\,\mathrm{d}\beta$. Thus for any ξ depending on a linear combination of independent variables one can obtain exact solutions of (8), but the extension of the class of such potentials ξ for which the technique described above can be applied will be the subject of a future article.

4. Solutions of the equation $R_{\boldsymbol{\alpha}}^{ps} \boldsymbol{u} = 0$

4.1.

We start with the simplest but necessary, case $\boldsymbol{\alpha} = 0$, $\alpha_0 = \gamma \in \mathbb{C}$. The constant γ can be included in the potential and if ξ satisfies the conditions (12) then the function $\xi + \gamma$ satisfies them also. The corresponding solution has the form

$$f = Q^{+}\mathrm{e}^{-\frac{(\xi+\gamma)^2}{2C}} a^{+} + Q^{-}\mathrm{e}^{\frac{(\xi+\gamma)^2}{2C}} a^{-} \in \ker(\boldsymbol{D} + (\xi+\gamma)I) \tag{14}$$

where Q^{+} and Q^{-} are the same projection operators defined by (13) and $\{a^{+}, a^{-}\} \subset \mathbb{H}(\mathbb{C})$.

For any $\alpha \in \mathbb{H}(\mathbb{C})$ we can include its scalar part α_0 in the potential. In this way we restrict our consideration to the most important case $\alpha = \boldsymbol{\alpha}$.

4.2. Let us assume that $\boldsymbol{\alpha}^2 \neq 0$. We denote by γ any complex square root from $\boldsymbol{\alpha}^2$: $\gamma^2 = \boldsymbol{\alpha}^2$. Let us consider the pair of mutually complementary projection operators

$$P^{\pm} := \frac{1}{2\gamma}\mathcal{M}^{(\gamma\pm\boldsymbol{\alpha})}.$$

The operator R_{α}^{ps} can be rewritten in the following form:

$$R_{\alpha}^{ps} = P^{+}(\boldsymbol{D} + (\xi+\gamma)I) + P^{-}(\boldsymbol{D} + (\xi-\gamma)I) \tag{15}$$

and $P^{\pm}$ commute with the operators in the parentheses. Consequently, the corresponding solution can be constructed with the aid of (14) and (15):

$$f = P^{+}(Q^{+}\mathrm{e}^{-\frac{(\xi+\gamma)^2}{2C}} a^{+} + Q^{-}\mathrm{e}^{\frac{(\xi+\gamma)^2}{2C}} a^{-}) + P^{-}(Q^{+}\mathrm{e}^{-\frac{(\xi-\gamma)^2}{2C}} b^{+} + Q^{-}\mathrm{e}^{\frac{(\xi-\gamma)^2}{2C}} b^{-}) \in \ker R_{\boldsymbol{\alpha}}^{ps} \tag{16}$$

$\boldsymbol{\alpha}^2 \neq 0$; $\{a^{+}, a^{-}, b^{+}, b^{-}\} \subset \mathbb{H}(\mathbb{C})$. Note that if $\boldsymbol{\alpha} = -(\mathrm{i}\omega i_1 + m i_2)$ (see section 2) then $\boldsymbol{\alpha}^2 \neq 0$ is equivalent to the inequality $\omega^2 \neq m^2$ because $\boldsymbol{\alpha}^2 = \omega^2 - m^2$ and the solution has the form

$$f = \frac{1}{2\sqrt{\omega^2 - m^2}}(Q^{+}\mathrm{e}^{-\frac{(\xi+\sqrt{\omega^2-m^2})^2}{2C}} a^{+} + Q^{-}\mathrm{e}^{\frac{(\xi+\sqrt{\omega^2-m^2})^2}{2C}} a^{-})(\sqrt{\omega^2 - m^2} - \mathrm{i}\omega i_1 - m i_2)$$
$$+(Q^{+}\mathrm{e}^{-\frac{(\xi-\sqrt{\omega^2-m^2})^2}{2C}} b^{+} + Q^{-}\mathrm{e}^{\frac{(\xi-\sqrt{\omega^2-m^2})^2}{2C}} b^{-})(\sqrt{\omega^2 - m^2} + \mathrm{i}\omega i_1 + m i_2). \tag{17}$$

As $P^{\pm}$ and $Q^{\pm}$ are projection operators (all pairwise commuting), the function (16) consists of four independent solutions of (7):

$$\begin{aligned} f^{++} &:= P^{+}(Q^{+}\mathrm{e}^{-\frac{(\xi+\gamma)^2}{2C}} a^{+}) \qquad & f^{+-} &:= P^{+}(Q^{-}\mathrm{e}^{\frac{(\xi+\gamma)^2}{2C}} a^{-}) \\ f^{-+} &:= P^{-}(Q^{+}\mathrm{e}^{-\frac{(\xi-\gamma)^2}{2C}} b^{+}) \qquad & f^{--} &:= P^{-}(Q^{-}\mathrm{e}^{\frac{(\xi-\gamma)^2}{2C}} b^{-}). \end{aligned} \tag{18}$$

4.2.

Finally we consider the case $\boldsymbol{\alpha}^2 = 0$ (for the complex quaternion $\boldsymbol{\alpha} = -(\mathrm{i}\omega i_1 + m i_2)$ this signifies that $\omega^2 = m^2$). If $v \in \ker(\boldsymbol{D} + \xi I)$ then the function $v\boldsymbol{\alpha} \in \ker(\boldsymbol{D} + \xi I + \mathcal{M}^{\boldsymbol{\alpha}})$ and the corresponding explicit solution of (7) can be represented as follows:

$$f = (Q^+ \mathrm{e}^{-\frac{\xi^2}{2C}} a^+ + Q^- \mathrm{e}^{\frac{\xi^2}{2C}} a^-)\boldsymbol{\alpha}. \tag{19}$$

5. Solutions of the Dirac equation with pseudoscalar potential

Now we are ready to construct a class of solutions to equation (4). We assume that the potential ϕ satisfies the following conditions in $G \subset \mathbb{R}^3$ (cf (12)):

$$(1)\ \Delta\phi = 0 \qquad (2)\ \langle \operatorname{grad}\phi, \operatorname{grad}\phi \rangle = C^2. \tag{20}$$

The equality (6) shows that if $f \in \ker R^{ps}_{\alpha}(\tilde{G})$ then $q := \mathcal{A}^{-1}[f] \in \ker \mathbb{D}^{ps}_{\omega,m}(G)$, where $\alpha = -(\mathrm{i}\omega i_1 + m i_2)$ and $\xi(x_1, x_2, x_3) = -\mathrm{i}\phi(x_1, x_2, -x_3)$. Thus, the only thing we have to do in order to obtain the solutions of (4) is to apply the transform $\mathcal{A}^{-1}$ to the solutions of (7) obtained in the previous section. Note that if ϕ satisfies (20), the corresponding potential ξ satisfies the conditions (12).

First, let us consider the case $\omega^2 \neq m^2$. The corresponding solutions in quaternionic terms were obtained in section 4.2. We rewrite the four independent solutions (18) in the component-wise form:

$$\begin{aligned} f^{++} = \frac{\mathrm{e}^{-\frac{(\xi+\gamma)^2}{2C}}}{4\gamma C}\{&(d_0^+ C + d_1^+ \partial_1\xi + d_2^+ \partial_2\xi + d_3^+ \partial_3\xi) i_0 \\ &+(-d_1^+ C + d_0^+ \partial_1\xi - d_3^+ \partial_2\xi + d_2^+ \partial_3\xi) i_1 \\ &+(-d_2^+ C + d_3^+ \partial_1\xi + d_0^+ \partial_2\xi - d_1^+ \partial_3\xi) i_2 \\ &+(-d_3^+ C - d_2^+ \partial_1\xi + d_1^+ \partial_2\xi + d_0^+ \partial_3\xi) i_3\} \end{aligned} \tag{21}$$

$$\begin{aligned} f^{+-} = \frac{\mathrm{e}^{\frac{(\xi+\gamma)^2}{2C}}}{4\gamma C}\{&(d_0^- C - d_1^- \partial_1\xi - d_2^- \partial_2\xi - d_3^- \partial_3\xi) i_0 \\ &+(-d_1^- C - d_0^- \partial_1\xi + d_3^- \partial_2\xi - d_2^- \partial_3\xi) i_1 \\ &+(-d_2^- C - d_3^- \partial_1\xi - d_0^- \partial_2\xi + d_1^- \partial_3\xi) i_2 \\ &+(-d_3^- C + d_2^- \partial_1\xi - d_1^- \partial_2\xi - d_0^- \partial_3\xi) i_3\} \end{aligned} \tag{22}$$

where

$$\begin{aligned} d_0^\pm &:= a_0^\pm \gamma + \mathrm{i}\omega a_1^\pm + m a_2^\pm \qquad & d_1^\pm &:= -a_1^\pm \gamma + \mathrm{i}\omega a_0^\pm - m a_3^\pm \\ d_2^\pm &:= -a_2^\pm \gamma + \mathrm{i}\omega a_3^\pm + m a_0^\pm \qquad & d_3^\pm &:= -a_3^\pm \gamma - \mathrm{i}\omega a_2^\pm + m a_1^\pm. \end{aligned}$$

Note that although the constants $\{a_0^\pm, a_1^\pm, a_2^\pm, a_3^\pm\}$ are independent, the introduced constants $\{d_0^\pm, d_1^\pm, d_2^\pm, d_3^\pm\}$ due to the action of projection operators are not already independent. A simple calculation shows that

$$\begin{aligned} &(\omega + m)(-d_0^\pm + \mathrm{i}d_3^\pm) = -\gamma(d_2^\pm - \mathrm{i}d_1^\pm) \\ &(\omega + m)(d_2^\pm + \mathrm{i}d_1^\pm) = -\gamma(d_0^\pm + \mathrm{i}d_3^\pm). \end{aligned} \tag{23}$$

By analogy we obtain the component-wise representations for f^{-+} and f^{--}:

$$f^{-+} = \frac{\mathrm{e}^{-\frac{(\xi-\gamma)^2}{2C}}}{4\gamma C}\{(l_0^+ C + l_1^+ \partial_1\xi + l_2^+ \partial_2\xi + l_3^+ \partial_3\xi) i_0$$

$$
\begin{aligned}
&+(-l_1^+C + l_0^+\partial_1\xi - l_3^+\partial_2\xi + l_2^+\partial_3\xi)i_1 \\
&+(-l_2^+C + l_3^+\partial_1\xi + l_0^+\partial_2\xi - l_1^+\partial_3\xi)i_2 \\
&+(-l_3^+C - l_2^+\partial_1\xi + l_1^+\partial_2\xi + l_0^+\partial_3\xi)i_3\}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
f^{--} = \frac{e^{\frac{(\xi-\gamma)^2}{2C}}}{4\gamma C}\{&(l_0^-C - l_1^-\partial_1\xi - l_2^-\partial_2\xi - l_3^-\partial_3\xi)i_0 \\
&+(-l_1^-C - l_0^-\partial_1\xi + l_3^-\partial_2\xi - l_2^-\partial_3\xi)i_1 \\
&+(-l_2^-C - l_3^-\partial_1\xi - l_0^-\partial_2\xi + l_1^-\partial_3\xi)i_2 \\
&+(-l_3^-C + l_2^-\partial_1\xi - l_1^-\partial_2\xi - l_0^-\partial_3\xi)i_3\}
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
l_0^\pm &:= b_0^\pm\gamma - \mathrm{i}\omega b_1^\pm - mb_2^\pm \qquad & l_1^\pm &:= -b_1^\pm\gamma - \mathrm{i}\omega b_0^\pm + mb_3^\pm \\
l_2^\pm &:= -b_2^\pm\gamma - \mathrm{i}\omega b_3^\pm - mb_0^\pm \qquad & l_3^\pm &:= -b_3^\pm\gamma + \mathrm{i}\omega b_2^\pm - mb_1^\pm
\end{aligned}
$$

and the following relations hold:

$$
\begin{aligned}
(\omega+m)(-l_0^\pm + \mathrm{i}l_3^\pm) &= \gamma(l_2^\pm - \mathrm{i}l_1^\pm) \\
(\omega+m)(l_2^\pm + \mathrm{i}l_1^\pm) &= \gamma(l_0^\pm + \mathrm{i}l_3^\pm).
\end{aligned} \tag{26}
$$

Now we are ready to apply the transform $\mathcal{A}^{-1}$ to the functions f^{++}, f^{+-}, f^{-+} and f^{--} in order to obtain four independent solutions of (4) $q^{++} := \mathcal{A}^{-1}[f^{++}]$, $q^{+-} := \mathcal{A}^{-1}[f^{+-}]$, $q^{-+} := \mathcal{A}^{-1}[f^{-+}]$ and $q^{--} := \mathcal{A}^{-1}[f^{--}]$. Taking into account that $\xi = -\mathrm{i}\tilde{\phi}$, $(\xi \pm \gamma)^2 = -(\tilde{\phi} \pm \mathrm{i}\gamma)^2$ and that $\partial_1\tilde{\phi} = \partial_1\phi$, $\partial_2\tilde{\phi} = \partial_2\phi$ and $\partial_3\tilde{\phi} = -\partial_3\phi$ we obtain

$$
q^{++} = \frac{e^{\frac{(\phi+\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix}
(d_2^+ + \mathrm{i}d_1^+)C + (-d_0^+ + \mathrm{i}d_3^+)\partial_1\phi + (d_3^+ + \mathrm{i}d_0^+)\partial_2\phi + (d_2^+ + \mathrm{i}d_1^+)\partial_3\phi \\
(-d_0^+ + \mathrm{i}d_3^+)C + (d_2^+ + \mathrm{i}d_1^+)\partial_1\phi + (-d_1^+ + \mathrm{i}d_2^+)\partial_2\phi + (d_0^+ - \mathrm{i}d_3^+)\partial_3\phi \\
(d_0^+ + \mathrm{i}d_3^+)C + (d_2^+ - \mathrm{i}d_1^+)\partial_1\phi + (-d_1^+ - \mathrm{i}d_2^+)\partial_2\phi + (d_0^+ + \mathrm{i}d_3^+)\partial_3\phi \\
(d_2^+ - \mathrm{i}d_1^+)C + (d_0^+ + \mathrm{i}d_3^+)\partial_1\phi + (-d_3^+ + \mathrm{i}d_0^+)\partial_2\phi + (-d_2^+ + \mathrm{i}d_1^+)\partial_3\phi
\end{pmatrix}
$$

$$
q^{+-} = \frac{e^{-\frac{(\phi+\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix}
(d_2^- + \mathrm{i}d_1^-)C - (-d_0^- + \mathrm{i}d_3^-)\partial_1\phi - (d_3^- + \mathrm{i}d_0^-)\partial_2\phi - (d_2^- + \mathrm{i}d_1^-)\partial_3\phi \\
(-d_0^- + \mathrm{i}d_3^-)C - (d_2^- + \mathrm{i}d_1^-)\partial_1\phi - (-d_1^- + \mathrm{i}d_2^-)\partial_2\phi - (d_0^- - \mathrm{i}d_3^-)\partial_3\phi \\
(d_0^- + \mathrm{i}d_3^-)C - (d_2^- - \mathrm{i}d_1^-)\partial_1\phi - (-d_1^- - \mathrm{i}d_2^-)\partial_2\phi - (d_0^- + \mathrm{i}d_3^-)\partial_3\phi \\
(d_2^- - \mathrm{i}d_1^-)C - (d_0^- + \mathrm{i}d_3^-)\partial_1\phi - (-d_3^- + \mathrm{i}d_0^-)\partial_2\phi - (-d_2^- + \mathrm{i}d_1^-)\partial_3\phi
\end{pmatrix}
$$

$$
q^{-+} = \frac{e^{\frac{(\phi-\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix}
(l_2^+ + \mathrm{i}l_1^+)C + (-l_0^+ + \mathrm{i}l_3^+)\partial_1\phi + (l_3^+ + \mathrm{i}l_0^+)\partial_2\phi + (l_2^+ + \mathrm{i}l_1^+)\partial_3\phi \\
(-l_0^+ + \mathrm{i}l_3^+)C + (l_2^+ + \mathrm{i}l_1^+)\partial_1\phi + (-l_1^+ + \mathrm{i}l_2^+)\partial_2\phi + (l_0^+ - \mathrm{i}l_3^+)\partial_3\phi \\
(l_0^+ + \mathrm{i}l_3^+)C + (l_2^+ - \mathrm{i}l_1^+)\partial_1\phi + (-l_1^+ - \mathrm{i}l_2^+)\partial_2\phi + (l_0^+ + \mathrm{i}l_3^+)\partial_3\phi \\
(l_2^+ - \mathrm{i}l_1^+)C + (l_0^+ + \mathrm{i}l_3^+)\partial_1\phi + (-l_3^+ + \mathrm{i}l_0^+)\partial_2\phi + (-l_2^+ + \mathrm{i}l_1^+)\partial_3\phi
\end{pmatrix}
$$

$$
q^{--} = \frac{e^{-\frac{(\phi-\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix}
(l_2^- + \mathrm{i}l_1^-)C - (-l_0^- + \mathrm{i}l_3^-)\partial_1\phi - (l_3^- + \mathrm{i}l_0^-)\partial_2\phi - (l_2^- + \mathrm{i}l_1^-)\partial_3\phi \\
(-l_0^- + \mathrm{i}l_3^-)C - (l_2^- + \mathrm{i}l_1^-)\partial_1\phi - (-l_1^- + \mathrm{i}l_2^-)\partial_2\phi - (l_0^- - \mathrm{i}l_3^-)\partial_3\phi \\
(l_0^- + \mathrm{i}l_3^-)C - (l_2^- - \mathrm{i}l_1^-)\partial_1\phi - (-l_1^- - \mathrm{i}l_2^-)\partial_2\phi - (l_0^- + \mathrm{i}l_3^-)\partial_3\phi \\
(l_2^- - \mathrm{i}l_1^-)C - (l_0^- + \mathrm{i}l_3^-)\partial_1\phi - (-l_3^- + \mathrm{i}l_0^-)\partial_2\phi - (-l_2^- + \mathrm{i}l_1^-)\partial_3\phi
\end{pmatrix}.
$$

Finally, let us introduce the following notations:

$$
\begin{aligned}
A_1^\pm &:= d_2^\pm + \mathrm{i}d_1^\pm \qquad & A_2^\pm &:= -d_0^\pm + \mathrm{i}d_3^\pm \\
B_1^\pm &:= l_2^\pm + \mathrm{i}l_1^\pm \qquad & B_2^\pm &:= -l_0^\pm + \mathrm{i}l_3^\pm.
\end{aligned}
$$

Then taking into account the relations (23), (26) we rewrite the obtained solutions in the following form:

$$
q^{++} = \frac{e^{\frac{(\phi+\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix}
q_0^{++} \\
q_1^{++} \\
-\frac{(\omega+m)}{\gamma}q_0^{++} \\
-\frac{(\omega+m)}{\gamma}q_1^{++}
\end{pmatrix} \tag{27}
$$

where

$$q_0^{++} := A_1^+ C + A_2^+ \partial_1\phi - \mathrm{i}A_2^+\partial_2\phi + A_1^+\partial_3\phi$$
$$q_1^{++} := A_2^+ C + A_1^+ \partial_1\phi + \mathrm{i}A_1^+\partial_2\phi - A_2^+\partial_3\phi$$
$$q^{+-} = \frac{\mathrm{e}^{-\frac{(\phi+\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix} q_0^{+-} \\ q_1^{+-} \\ -\frac{(\omega+m)}{\gamma}q_0^{+-} \\ -\frac{(\omega+m)}{\gamma}q_1^{+-}\end{pmatrix} \tag{28}$$

where

$$q_0^{+-} := A_1^- C - A_2^- \partial_1\phi + \mathrm{i}A_2^-\partial_2\phi - A_1^-\partial_3\phi$$
$$q_1^{+-} := A_2^- C - A_1^- \partial_1\phi - \mathrm{i}A_1^-\partial_2\phi + A_2^-\partial_3\phi$$
$$q^{-+} = \frac{\mathrm{e}^{\frac{(\phi-\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix} q_0^{-+} \\ q_1^{-+} \\ \frac{(\omega+m)}{\gamma}q_0^{-+} \\ \frac{(\omega+m)}{\gamma}q_1^{-+}\end{pmatrix} \tag{29}$$

where

$$q_0^{-+} := B_1^+ C + B_2^+ \partial_1\phi - \mathrm{i}B_2^+\partial_2\phi + B_1^+\partial_3\phi$$
$$q_1^{-+} := B_2^+ C + B_1^+ \partial_1\phi + \mathrm{i}B_1^+\partial_2\phi - B_2^+\partial_3\phi$$
$$q^{--} = \frac{\mathrm{e}^{-\frac{(\phi-\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix} q_0^{--} \\ q_1^{--} \\ \frac{(\omega+m)}{\gamma}q_0^{--} \\ \frac{(\omega+m)}{\gamma}q_1^{--}\end{pmatrix} \tag{30}$$

where

$$q_0^{--} := B_1^- C - B_2^- \partial_1\phi + \mathrm{i}B_2^-\partial_2\phi - B_1^-\partial_3\phi$$
$$q_1^{--} := B_2^- C - B_1^- \partial_1\phi - \mathrm{i}B_1^-\partial_2\phi + B_2^-\partial_3\phi.$$

Thus, we obtain the following statement.

Theorem 2. The functions defined by (27)–(30), where $A_1^\pm$, $A_2^\pm$, $B_1^\pm$ and $B_2^\pm$ are arbitrary (independent) complex constants, belong to $\ker \mathbb{D}_{\omega,m}^{ps}$ when the potential ϕ satisfies (20) and $\omega^2 \neq m^2$.

To verify this statement one can simply substitute any of the functions q^{++}, q^{+-}, q^{-+}, q^{--} into equation (4) using its explicit form:

$$\partial_1 q_3 - \mathrm{i}\partial_2 q_3 + \partial_3 q_2 - \phi q_2 + \mathrm{i}(\omega+m)q_0 = 0$$
$$\partial_1 q_2 + \mathrm{i}\partial_2 q_2 - \partial_3 q_3 - \phi q_3 + \mathrm{i}(\omega+m)q_1 = 0$$
$$-\partial_1 q_1 + \mathrm{i}\partial_2 q_1 - \partial_3 q_0 + \phi q_0 - \mathrm{i}(\omega-m)q_2 = 0$$
$$-\partial_1 q_0 - \mathrm{i}\partial_2 q_0 + \partial_3 q_1 + \phi q_1 - \mathrm{i}(\omega-m)q_3 = 0.$$

Let us consider the following example: $\phi(\boldsymbol{x}) = ax_1 + bx_2 + cx_3$, $\{a, b, c\} \subset \mathbb{R}$ then $C = \sqrt{a^2+b^2+c^2}$, $\gamma = \sqrt{\omega^2 - m^2}$. Let us analyse, for instance, the function (28) which takes the form

$$q^{+-} = \frac{\mathrm{e}^{-\frac{(ax_1+bx_2+cx_3+\mathrm{i}\gamma)^2}{2C}}}{4\gamma C}\begin{pmatrix} A_1^-(C-c) - A_2^-(a-\mathrm{i}b) \\ A_2^-(C+c) - A_1^-(a+\mathrm{i}b) \\ -\frac{(\omega+m)}{\gamma}(A_1^-(C-c) - A_2^-(a-\mathrm{i}b)) \\ -\frac{(\omega+m)}{\gamma}(A_2^-(C+c) - A_1^-(a+\mathrm{i}b))\end{pmatrix}$$

where A_1^- and A_2^- are arbitrary complex constants. Notice that the function q^{+-} belongs to $\ker \mathbb{D}^{ps}_{\omega,m}(\mathbb{R}^3)$ for any $\omega \in \mathbb{R}$ such that $\omega^2 \neq m^2$, but that if we also require that q^{+-} belong to the Sobolev space $H^1(\mathbb{R}^3)$ (see, e.g. [20, p 253]) then ω must satisfy the condition $\omega^2 < m^2$.

Now consider the case $\omega^2 = m^2$. The corresponding solution in the quaternionic form is defined by (19). Let us introduce the following notations for two independent parts of the function (19):

$$f^+ := Q^+ \mathrm{e}^{-\frac{\xi^2}{2C}} a^+ \boldsymbol{\alpha} \qquad f^- := Q^- \mathrm{e}^{\frac{\xi^2}{2C}} a^- \boldsymbol{\alpha}.$$

In the explicit form we have

$$\begin{aligned}
f^+ = \frac{\mathrm{e}^{-\frac{\xi^2}{2C}}}{2C}\{&((\mathrm{i}\omega a_1^+ + m a_2^+)C + (\mathrm{i}\omega a_0^+ - m a_3^+)\partial_1\xi + (\mathrm{i}\omega a_3^+ m a_0^+)\partial_2\xi \\
&+ (-\mathrm{i}\omega a_2^+ + m a_1^+)\partial_3\xi)i_0 + (-(\mathrm{i}\omega a_0^+ - m a_3^+)C + (\mathrm{i}\omega a_1^+ + m a_2^+)\partial_1\xi \\
&- (-\mathrm{i}\omega a_2^+ + m a_1^+)\partial_2\xi + (\mathrm{i}\omega a_3^+ m a_0^+)\partial_3\xi)i_1 \\
&+ (-(\mathrm{i}\omega a_3^+ m a_0^+)C + (-\mathrm{i}\omega a_2^+ + m a_1^+)\partial_1\xi + (\mathrm{i}\omega a_1^+ + m a_2^+)\partial_2\xi \\
&- (\mathrm{i}\omega a_0^+ - m a_3^+)\partial_3\xi)i_2 + (-(-\mathrm{i}\omega a_2^+ + m a_1^+)C - (\mathrm{i}\omega a_3^+ m a_0^+)\partial_1\xi \\
&+ (\mathrm{i}\omega a_0^+ - m a_3^+)\partial_2\xi + (\mathrm{i}\omega a_1^+ + m a_2^+)\partial_3\xi)i_3\}
\end{aligned}$$

$$\begin{aligned}
f^- = \frac{\mathrm{e}^{\frac{\xi^2}{2C}}}{2C}\{&((\mathrm{i}\omega a_1^- + m a_2^-)C - (\mathrm{i}\omega a_0^- - m a_3^-)\partial_1\xi - (\mathrm{i}\omega a_3^- m a_0^-)\partial_2\xi \\
&- (-\mathrm{i}\omega a_2^- + m a_1^-)\partial_3\xi)i_0 + (-(\mathrm{i}\omega a_0^- - m a_3^-)C - (\mathrm{i}\omega a_1^- + m a_2^-)\partial_1\xi \\
&+ (-\mathrm{i}\omega a_2^- + m a_1^-)\partial_2\xi - (\mathrm{i}\omega a_3^- m a_0^-)\partial_3\xi)i_1 \\
&+ (-(\mathrm{i}\omega a_3^- m a_0^-)C - (-\mathrm{i}\omega a_2^- + m a_1^-)\partial_1\xi - (\mathrm{i}\omega a_1^- + m a_2^-)\partial_2\xi \\
&+ (\mathrm{i}\omega a_0^- - m a_3^-)\partial_3\xi)i_2 + (-(-\mathrm{i}\omega a_2^- + m a_1^-)C + (\mathrm{i}\omega a_3^- m a_0^-)\partial_1\xi \\
&- (\mathrm{i}\omega a_0^- - m a_3^-)\partial_2\xi - (\mathrm{i}\omega a_1^- + m a_2^-)\partial_3\xi)i_3\}.
\end{aligned}$$

Applying the transform $\mathcal{A}^{-1}$ to the functions f^+ and f^- we obtain two independent solutions of (4):

$$q^+ := \mathcal{A}^{-1}[f^+] = \frac{\mathrm{e}^{\frac{\phi^2}{2C}}}{2C} \times \begin{pmatrix} (\omega - m)((-a_0^+ + \mathrm{i}a_3^+)C + (a_2^+ - \mathrm{i}a_1^+)\partial_1\phi - \mathrm{i}(a_2^+ - \mathrm{i}a_1^+)\partial_2\phi + (-a_0^+ + \mathrm{i}a_3^+)\partial_3\phi) \\ (\omega - m)((a_2^+ - \mathrm{i}a_1^+)C + (-a_0^+ + \mathrm{i}a_3^+)\partial_1\phi + \mathrm{i}(-a_0^+ + \mathrm{i}a_3^+)\partial_2\phi - (a_2^+ - \mathrm{i}a_1^+)\partial_3\phi) \\ (\omega + m)((a_2^+ + \mathrm{i}a_1^+)C + (a_0^+ + \mathrm{i}a_3^+)\partial_1\phi - \mathrm{i}(a_0^+ + \mathrm{i}a_3^+)\partial_2\phi + (a_2^+ + \mathrm{i}a_1^+)\partial_3\phi) \\ (\omega + m)((a_0^+ + \mathrm{i}a_3^+)C + (a_2^+ + \mathrm{i}a_1^+)\partial_1\phi + \mathrm{i}(a_2^+ + \mathrm{i}a_1^+)\partial_2\phi - (a_0^+ + \mathrm{i}a_3^+)\partial_3\phi) \end{pmatrix}$$

$$q^- := \mathcal{A}^{-1}[f^-] = \frac{\mathrm{e}^{-\frac{\phi^2}{2C}}}{2C} \times \begin{pmatrix} (\omega - m)((-a_0^- + \mathrm{i}a_3^-)C - (a_2^- - \mathrm{i}a_1^-)\partial_1\phi + \mathrm{i}(a_2^- - \mathrm{i}a_1^-)\partial_2\phi - (-a_0^- + \mathrm{i}a_3^-)\partial_3\phi) \\ (\omega - m)((a_2^- - \mathrm{i}a_1^-)C - (-a_0^- + \mathrm{i}a_3^-)\partial_1\phi - \mathrm{i}(-a_0^- + \mathrm{i}a_3^-)\partial_2\phi - (a_2^- - \mathrm{i}a_1^-)\partial_3\phi) \\ (\omega + m)((a_2^- + \mathrm{i}a_1^-)C - (a_0^- + \mathrm{i}a_3^-)\partial_1\phi + \mathrm{i}(a_0^- + \mathrm{i}a_3^-)\partial_2\phi - (a_2^- + \mathrm{i}a_1^-)\partial_3\phi) \\ (\omega + m)((a_0^- + \mathrm{i}a_3^-)C - (a_2^- + \mathrm{i}a_1^-)\partial_1\phi - \mathrm{i}(a_2^- + \mathrm{i}a_1^-)\partial_2\phi + (a_0^- + \mathrm{i}a_3^-)\partial_3\phi) \end{pmatrix}.$$

Note that for both cases $\omega = m$ and $\omega = -m$ we obtain two-component solutions. When $\omega = m$ the two first components of each of the solutions q^+ and q^- are zero. Similarly, when $\omega = -m$ the last two components are zero.

Thus, the following statement is true.

Theorem 3. The functions q^+ and q^-, where $a_k^{\pm}$, $k = \overline{0,3}$ are arbitrary (independent) complex constants, belong to $\ker \mathbb{D}^{ps}_{\omega,m}$ when the potential ϕ satisfies (20) and $\omega^2 = m^2$.

Note that to obtain the corresponding solutions of (1) one has just to multiply the functions from $\ker \mathbb{D}^{ps}_{\omega,m}$ by $\mathrm{e}^{\mathrm{i}\omega t}$.

6. Solutions for the scalar potential

In this section we find some exact solutions to the following Dirac equation with scalar potential:

$$\mathbb{D}^{sc}_m q(\boldsymbol{x}) := \left[-\sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}m + \phi(\boldsymbol{x}) \right] q(\boldsymbol{x}) = 0. \tag{31}$$

The scalar potential (see, e.g. [22, p 108]) may be considered as an $\boldsymbol{x}$-dependent rest mass and is used as a model for quark confinement. We assume that ϕ satisfies (12). Applying the transform $\mathcal{A}$ we obtain that

$$R^{sc}_\alpha = -\mathcal{A}\gamma_1\gamma_2\gamma_3\mathbb{D}^{sc}_m\mathcal{A}^{-1}$$

where

$$R^{sc}_\alpha := \boldsymbol{D} + \mathcal{M}^{\mathrm{i}\tilde{\phi}(\boldsymbol{x})i_2} + \mathcal{M}^\alpha \alpha = -mi_2.$$

Let us introduce the following projection operators

$$Z^{\pm} := \tfrac{1}{2}\mathcal{M}^{(1\pm \mathrm{i}i_2)}.$$

Then the operator R^{sc}_α can be rewritten in the following form

$$R^{sc}_\alpha = Z^+(\boldsymbol{D} + (\tilde{\phi} + \mathrm{i}m)I) + Z^-(\boldsymbol{D} - (\tilde{\phi} + \mathrm{i}m)I)$$

where the operators $Z^{\pm}$ commute with the operators in parentheses. Consequently, we may look for the solutions to (31) of the form

$$u = Z^+ f + Z^- g$$

where $f \in \ker(\boldsymbol{D} + (\tilde{\phi} + \mathrm{i}m)I)$ and $g \in \ker(\boldsymbol{D} - (\tilde{\phi} + \mathrm{i}m)I)$. Thus, we can use the results of section 4.1 and immediately write down the corresponding function $u \in \ker R^{sc}_\alpha$:

$$u = Z^+(Q^+\mathrm{e}^{-\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}a^+ + Q^-\mathrm{e}^{\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}a^-) + Z^-(Q^-\mathrm{e}^{-\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}b^+ + Q^+\mathrm{e}^{\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}b^-)$$

where $Q^{\pm} := \frac{1}{2}(1 \pm \frac{1}{C}\,\mathrm{grad}\,\tilde{\phi})$ and $a^{\pm}$, $b^{\pm}$ are arbitrary constant complex quaternions.

Let us introduce the following notations:

$$\begin{aligned} u^{++} &:= Z^+(Q^+\mathrm{e}^{-\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}a^+) \qquad & u^{+-} &:= Z^+(Q^-\mathrm{e}^{\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}a^-) \\ u^{-+} &:= Z^-(Q^-\mathrm{e}^{-\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}b^+) \qquad & u^{--} &:= Z^-(Q^+\mathrm{e}^{\frac{(\tilde{\phi}+\mathrm{i}m)^2}{2C}}b^-). \end{aligned} \tag{32}$$

Now let us apply the transform $\mathcal{A}^{-1}$ to the functions u^{++}, u^{+-}, u^{-+}, u^{--}. Then introducing the following notations:

$$\begin{aligned} A^{\pm}_0 &:= -\tfrac{1}{2}(\mathrm{i}a^{\pm}_0 + \mathrm{i}a^{\pm}_1 + a^{\pm}_2 + a^{\pm}_3) \qquad & A^{\pm}_1 &:= \tfrac{1}{2}(-\mathrm{i}a^{\pm}_0 + \mathrm{i}a^{\pm}_1 - a^{\pm}_2 + a^{\pm}_3) \\ B^{\pm}_0 &:= \tfrac{1}{2}(\mathrm{i}b^{\pm}_0 - \mathrm{i}b^{\pm}_1 - b^{\pm}_2 + b^{\pm}_3) \qquad & B^{\pm}_1 &:= \tfrac{1}{2}(-\mathrm{i}b^{\pm}_0 - \mathrm{i}b^{\pm}_1 + b^{\pm}_2 + b^{\pm}_3) \end{aligned}$$

we obtain four independent solutions to (31):

$$q^{++} := \mathcal{A}^{-1}[u^{++}] = \frac{\mathrm{e}^{-\frac{(\phi+\mathrm{i}m)^2}{2C}}}{2C} \begin{pmatrix} q^{++}_0 \\ q^{++}_1 \\ \mathrm{i}q^{++}_0 \\ \mathrm{i}q^{++}_1 \end{pmatrix} \tag{33}$$

where

$$q_0^{++} := A_0^+ C + A_1^+ \partial_1 \phi - \mathrm{i} A_1^+ \partial_2 \phi + \mathrm{i} A_0^+ \partial_3 \phi$$
$$q_1^{++} := -\mathrm{i} A_1^+ C + \mathrm{i} A_0^+ \partial_1 \phi - A_0^+ \partial_2 \phi - A_1^+ \partial_3 \phi$$
$$q^{+-} := \mathcal{A}^{-1}[u^{+-}] = \frac{\mathrm{e}^{\frac{(\phi + \mathrm{i}m)^2}{2C}}}{2C} \begin{pmatrix} q_0^{+-} \\ q_1^{+-} \\ \mathrm{i} q_0^{+-} \\ \mathrm{i} q_1^{+-} \end{pmatrix} \tag{34}$$

where

$$q_0^{+-} := A_0^- C - A_1^- \partial_1 \phi + \mathrm{i} A_1^- \partial_2 \phi - \mathrm{i} A_0^- \partial_3 \phi$$
$$q_1^{+-} := -\mathrm{i} A_1^- C - \mathrm{i} A_0^- \partial_1 \phi + A_0^- \partial_2 \phi + A_1^- \partial_3 \phi$$
$$q^{-+} := \mathcal{A}^{-1}[u^{-+}] = \frac{\mathrm{e}^{-\frac{(\phi + \mathrm{i}m)^2}{2C}}}{2C} \begin{pmatrix} q_0^{-+} \\ q_1^{-+} \\ -\mathrm{i} q_0^{-+} \\ -\mathrm{i} q_1^{-+} \end{pmatrix} \tag{35}$$

where

$$q_0^{-+} := B_0^+ C - B_1^+ \partial_1 \phi + \mathrm{i} B_1^+ \partial_2 \phi - \mathrm{i} B_0^+ \partial_3 \phi$$
$$q_1^{-+} := -\mathrm{i} B_1^+ C - \mathrm{i} B_0^+ \partial_1 \phi + B_0^+ \partial_2 \phi + B_1^+ \partial_3 \phi$$
$$q^{--} := \mathcal{A}^{-1}[u^{--}] = \frac{\mathrm{e}^{\frac{(\phi + \mathrm{i}m)^2}{2C}}}{2C} \begin{pmatrix} q_0^{--} \\ q_1^{--} \\ -\mathrm{i} q_0^{--} \\ -\mathrm{i} q_1^{--} \end{pmatrix} \tag{36}$$

where

$$q_0^{--} := B_0^- C + B_1^- \partial_1 \phi - \mathrm{i} B_1^- \partial_2 \phi + \mathrm{i} B_0^- \partial_3 \phi$$
$$q_1^{--} := -\mathrm{i} B_1^- C + \mathrm{i} B_0^- \partial_1 \phi - B_0^- \partial_2 \phi - B_1^- \partial_3 \phi.$$

Let us formulate this result as follows.

Theorem 4. The functions (33)–(36), where $A_0^\pm$, $A_1^\pm$, $B_0^\pm$, $B_1^\pm$ are arbitrary complex constants, belong to $\ker \mathbb{D}_m^{sc}$ when the potential ϕ satisfies (12).

7. Solutions for the electric potential

We consider the time-harmonic solutions $\Phi(t, \boldsymbol{x}) = q(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\omega t}$ to the massless Dirac equation with electric potential:

$$\left[\gamma_0 \partial_t - \sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}\gamma_0 \phi(\boldsymbol{x})\right] \Phi(t, \boldsymbol{x}) = 0. \tag{37}$$

For q we have the following equation:

$$\mathbb{D}_\omega^{el} q(\boldsymbol{x}) := \left[\mathrm{i}\omega\gamma_0 - \sum_{k=1}^{3} \gamma_k \partial_k + \mathrm{i}\gamma_0 \phi(\boldsymbol{x})\right] q(\boldsymbol{x}) = 0. \tag{38}$$

Here the electric potential ϕ satisfies (12). Applying the transform $\mathcal{A}$ we obtain the equality

$$R_\alpha^{el} = -\mathcal{A}\gamma_1\gamma_2\gamma_3 \mathbb{D}_\omega^{el} \mathcal{A}^{-1}$$

where

$$R_\alpha^{el} := \boldsymbol{D} + \mathcal{M}^{-\mathrm{i}\tilde{\phi}(\boldsymbol{x}) i_1} + \mathcal{M}^\alpha$$

$\alpha = -\mathrm{i}\omega i_1$.

Let us introduce the following projection operators:

$$S^{\pm} := \tfrac{1}{2}\mathcal{M}^{(1\mp \mathrm{i}i_1)}.$$

Then a simple calculation shows that

$$R_\alpha^{el} = S^+(\boldsymbol{D} + (\tilde{\phi}(\boldsymbol{x}) + \omega)I) + S^-(\boldsymbol{D} - (\tilde{\phi}(\boldsymbol{x}) + \omega)I)$$

where the operators $S^{\pm}$ commute with the operators in parentheses. Thus we reduce the problem to the case considered in section 4.1, because any two functions $f \in \ker(\boldsymbol{D} + (\tilde{\phi}(\boldsymbol{x}) + \omega)I)$ and $g \in \ker(\boldsymbol{D} - (\tilde{\phi}(\boldsymbol{x}) + \omega)I)$ give us a function u from $\ker R_\alpha^{el}$:

$$u = S^+ f + S^- g \in \ker R_\alpha^{el}.$$

Using functions of the form (14) we obtain the corresponding solutions for the operator R_α^{el}:

$$u = S^+(Q^+\mathrm{e}^{-\frac{(\tilde{\phi}+\omega)^2}{2C}}a^+ + Q^-\mathrm{e}^{\frac{(\tilde{\phi}+\omega)^2}{2C}}a^-) + S^-(Q^-\mathrm{e}^{-\frac{(\tilde{\phi}+\omega)^2}{2C}}b^+ + Q^+\mathrm{e}^{\frac{(\tilde{\phi}+\omega)^2}{2C}}b^-). \quad (39)$$

The operators $Q^{\pm}$ are defined by the equalities $Q^{\pm} := \frac{1}{2}(1 \pm \frac{1}{C}\,\mathrm{grad}\,\tilde{\phi})I$ and $a^{\pm}$, $b^{\pm}$ are arbitrary constant complex quaternions.

Let us introduce the following notations:

$$\begin{aligned} u^{++} &:= S^+(Q^+\mathrm{e}^{-\frac{(\tilde{\phi}+\omega)^2}{2C}}a^+) \qquad & u^{+-} &:= S^+(Q^-\mathrm{e}^{\frac{(\tilde{\phi}+\omega)^2}{2C}}a^-) \\ u^{-+} &:= S^-(Q^-\mathrm{e}^{-\frac{(\tilde{\phi}+\omega)^2}{2C}}b^+) \qquad & u^{--} &:= S^-(Q^+\mathrm{e}^{\frac{(\tilde{\phi}+\omega)^2}{2C}}b^-). \end{aligned} \quad (40)$$

Applying the transform $\mathcal{A}^{-1}$ to the functions (40) and introducing the notations

$$\begin{aligned} A_0^{\pm} &:= \tfrac{1}{2}(-a_0^{\pm} - \mathrm{i}a_1^{\pm} - a_2^{\pm} + \mathrm{i}a_3^{\pm}) \qquad & A_1^{\pm} &:= \tfrac{1}{2}(-\mathrm{i}a_0^{\pm} + a_1^{\pm} + \mathrm{i}a_2^{\pm} + a_3^{\pm}) \\ B_0^{\pm} &:= \tfrac{1}{2}(b_0^{\pm} - \mathrm{i}b_1^{\pm} - b_2^{\pm} - \mathrm{i}b_3^{\pm}) \qquad & B_1^{\pm} &:= \tfrac{1}{2}(\mathrm{i}b_0^{\pm} + b_1^{\pm} + \mathrm{i}b_2^{\pm} - b_3^{\pm}) \end{aligned}$$

we obtain the following four independent solutions to (38):

$$q^{++} := \mathcal{A}^{-1}[u^{++}] = \frac{\mathrm{e}^{-\frac{(\phi+\omega)^2}{2C}}}{2C}\begin{pmatrix} q_0^{++} \\ q_1^{++} \\ -q_0^{++} \\ -q_1^{++} \end{pmatrix} \quad (41)$$

where

$$\begin{aligned} q_0^{++} &:= A_0^+C + A_1^+\partial_1\phi - \mathrm{i}A_1^+\partial_2\phi + \mathrm{i}A_0^+\partial_3\phi \\ q_1^{++} &:= -\mathrm{i}A_1^+C + \mathrm{i}A_0^+\partial_1\phi - A_0^+\partial_2\phi - A_1^+\partial_3\phi \end{aligned}$$

$$q^{+-} := \mathcal{A}^{-1}[u^{+-}] = \frac{\mathrm{e}^{\frac{(\phi+\omega)^2}{2C}}}{2C}\begin{pmatrix} q_0^{+-} \\ q_1^{+-} \\ -q_0^{+-} \\ -q_1^{+-} \end{pmatrix} \quad (42)$$

where

$$\begin{aligned} q_0^{+-} &:= A_0^-C - A_1^-\partial_1\phi + \mathrm{i}A_1^-\partial_2\phi - \mathrm{i}A_0^-\partial_3\phi \\ q_1^{+-} &:= -\mathrm{i}A_1^-C - \mathrm{i}A_0^-\partial_1\phi + A_0^-\partial_2\phi + A_1^-\partial_3\phi \end{aligned}$$

$$q^{-+} := \mathcal{A}^{-1}[u^{-+}] = \frac{\mathrm{e}^{-\frac{(\phi+\omega)^2}{2C}}}{2C}\begin{pmatrix} q_0^{-+} \\ q_1^{-+} \\ q_0^{-+} \\ q_1^{-+} \end{pmatrix} \quad (43)$$

where

$$q_0^{-+} := B_0^+ C + B_1^+ \partial_1\phi - \mathrm{i}B_1^+ \partial_2\phi - \mathrm{i}B_0^+ \partial_3\phi$$
$$q_1^{-+} := \mathrm{i}B_1^+ C - \mathrm{i}B_0^+ \partial_1\phi + B_0^+ \partial_2\phi - B_1^+ \partial_3\phi$$
$$q^{--} := \mathcal{A}^{-1}[u^{--}] = \frac{\mathrm{e}^{\frac{(\phi+\omega)^2}{2C}}}{2C}\begin{pmatrix} q_0^{--} \\ q_1^{--} \\ q_0^{--} \\ q_1^{--} \end{pmatrix} \qquad (44)$$

where

$$q_0^{--} := B_0^- C - B_1^- \partial_1\phi + \mathrm{i}B_1^- \partial_2\phi + \mathrm{i}B_0^- \partial_3\phi$$
$$q_1^{--} := \mathrm{i}B_1^- C + \mathrm{i}B_0^- \partial_1\phi - B_0^- \partial_2\phi + B_1^- \partial_3\phi.$$

Thus, we proved the following statement.

Theorem 5. The functions (41)–(44), where $A_0^\pm$, $A_1^\pm$, $B_0^\pm$, $B_1^\pm$ are arbitrary complex constants, belong to $\ker \mathbb{D}_\omega^{el}$ when the potential ϕ satisfies (12).

The solutions of (3) (for $m = 0$) may be obtained from the functions (41)–(44) by multiplying by $\mathrm{e}^{\mathrm{i}\omega t}$.

Remark 2. Note that if ϕ is a linear combination of independent variables then the corresponding electric field is constant and some exact solutions for this case (containing Airy functions) were obtained using the technique of separation of variables (see, e.g. [1]). Solutions (41)–(44) were obtained by a completely different technique which is why they do not coincide with the known ones and have a simpler form.

8. Conclusions

The technique of biquaternionic projection operators shown in action in this work allowed us to obtain some families of solutions of Dirac's equation with a harmonic one-component potential the gradient squared of which is a constant. The last condition is quite restrictive but an interesting point is that the method of separation of variables was always the main tool for obtaining exact solutions of relativistic wave equations and here it is substituted by an essentially different technique which takes into account other characteristics of the potential. That is why the class of potentials considered in this work is so different from the potentials for which there were known exact solutions. The same can be said about spectrum theory of Dirac's operator. Usually, all conclusions about the spectrum are based on the behaviour of the potential at infinity or other asymptotic properties. In this work we obtained information on the spectrum proceeding from differential characteristics of the potential. Finally, the technique proposed in this paper is able to give results for a much wider class of potentials, at least for potentials described in remark 1, and so this work will be continued.

Acknowledgment

The author is very grateful to Professor Vladimir V Belov for helpful and stimulating discussions (during a stay at the Department of Applied Mathematics, Moscow State Institute of Electronics and Mathematics) that resulted in this work.

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